Journal of Statistical Physics, Vol. 82, Nos. 1/2, 1996

On the Multifractal Analysis of Bernoulli Convolutions. I. Large-Deviation Results

François Ledrappier[1] and Anna Porzio[2]

Received January 20, 1995; final May 22, 1995

We show how the formalism developed in a previous paper allows us to exhibit the multifractal nature of the infinitely convolved Bernoulli measures ν_γ, for γ the golden mean. In this first part we establish some large-deviation results for random products of matrices, using perturbation theory of quasicompact operators.

KEY WORDS: Random matrices; large deviations.

1. INTRODUCTION

1.1. Problem I. The Singularity of the Golden Mean Bernoulli Convolution

Let $\varepsilon_1, \varepsilon_2,...$ be a sequence of independent random variables each taking the values $+1$ and -1 with equal probability. The probability distribution of the random variable $(1-\gamma)\sum_{n=0}^{\infty} \varepsilon_n \gamma^n$, $0<\gamma<1$, defines a measure ν_γ which is called an infinitely convolved Bernoulli measure or simply a Bernoulli convolution. For $\gamma > 1/2$ it is a difficult, old, and not yet completely solved problem to decide on the nature of ν_γ.[13,14,10,11] Recently Solomiak[32] proved that for almost all $\gamma \in [1/2, 1]$, ν_γ is absolutely continuous, improving a result of Erdös.[10] The work of Alexander and

[1] Laboratoire de Probabilités, Université de Paris VI, 75005 Paris, and École Polytechnique, CMAT, 91128 Palaiseau, France (CNRS URA 169). E-mail: Ledrappier@orphee.polytechnique.fr.

[2] Université Paris XIII, 93430 Villetaneuse, and CPTH École Polytechnique 91128 Palaiseau, France. E-mail: Porzio@orphee.polytechnique.fr.

0022-4715/96/0100-0367$09.50/0

Yorke[1] relates to dynamics this old arithmetic measure problem. They consider the map $(x, y) \in (-\infty, +\infty) \times [-1, +1] \to T_\gamma(x, y)$:

$$T_\gamma(x, y) = \begin{cases} \gamma x + 1 - \gamma, & 2y - 1 \quad \text{if} \quad y \geqslant 0 \\ \gamma x - (1 - \gamma), & 2y + 1 \quad \text{if} \quad y < 0 \end{cases} \tag{1}$$

For $1/2 < \gamma < 1$, T_γ is the "fat" baker's transformation: the map is *not* invertible, the attractor is the whole square $[-1, +1] \times [-1, +1]$, and it possesses a Sinai–Bowen–Ruelle measure whose transverse component is ν_γ. Recall that the Hausdorff dimension (HD) of a Borel probability measure μ on a compact metric space M is the HD of the smallest set of full measure: $HD(\mu) = \inf\{HD(Y), Y: \mu(Y) = 1, Y \subset M\}$. Young[33] proved that if μ is a Borel probability measure on a compact Riemannian manifold, and *if* μ a.e.

$$\lim_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} = \alpha \tag{2}$$

$[B_\varepsilon(x)$ is an ε-ball centered in $x]$, then $HD(\mu) = \alpha$. In the dynamical system context (1) the limit exists.[33,24,23] Alexander and Yorke, relying on the work of Garsia,[13,14] found numerically the value $HD(\nu_\gamma)$ for $\gamma =$ golden mean. Alexander and Zagier[2] and Lalley[19] have a theoretical entropy formula for $\beta =$ golden mean which agrees with this[1] empirical result. Bovier[5] has another proof of the singularity of ν_γ in that case.

In ref. 22 we described in great detail the invariant measure of the fat baker's transformation, and we gave an explicit (i.e., numerically computable) theoretical formula for the dimension of ν_γ in this nice case. Our approach is dynamical; we introduced ν_γ as the transverse measure of the maximum entropy measure μ on the repelling set invariant for the contracting maps of the square $T_0^{-1} = (\gamma x, y/2)$ and $T_1^{-1} = (\gamma x + 1 - \gamma, (y + 1)/2)$. By refs. 24 and 23 we know that ν_γ always satisfies (2), so that all notions of dimension coincide. Our approach strongly relies on the Markov structure of the two-dimensional system: indeed, if $\gamma =$ golden mean, the fat baker's transformation has a very simple Markov coding. The "ambiguity" (of order two) of this coding, which appears when projecting on the line, due to passagers for the central, overlapping zone, can be expressed by means of products of matrices (or order two). This product has a Markov distribution inherited by the Markov structure of the underlying dynamical system. The dimension of the projected measure is therefore associated with the growth of this product; our dimension formula appears in a natural way as a version of the Furstenberg–Guivarch formula.[12,10] The result of Young (2) ensures that this quantity gives actually the (information) dimension of the measure. Observe that there are other random products of matrices which might naturally occur in this problem (R. Kenyon, Y. Peres, and S. Lalley,[19] private communications).

We summarize in the following items the formalism and the results of ref. 22.

(a) The setting. We considered the (noninvertible) map $(x, y) \to T(x, y)$:

$$T(x,y)=\begin{cases}\dfrac{x}{y}, & 2y \quad \text{if} \quad y \leqslant 1/2, \quad x \leqslant \gamma \\ \dfrac{x}{y}-\gamma, & 2y-1 \quad \text{if} \quad y \geqslant 1/2, \quad x \geqslant 1-\beta\end{cases} \tag{3}$$

with $\gamma+\gamma^2=1$. Let $A=[1-\gamma, \gamma]\times[1/2, 1]$, $B=[\gamma, 1]\times[1/2, 1]$, $C=[0, 1-\gamma]\times[0, 1/2]$, $D=[1-\gamma, \gamma]\times[0, 1/2]$. Since $\gamma+\gamma^2=1$, $\{A, B, C, D\}$ is a Markov partition with compatibility rules: $A \to C$; $B \to A$, B, D; $C \to A$, C, D; $D \to B$. That is, every point $(x, y)\in X$ is coded by a sequence $\mathbf{a}(x, y)=a_0a_1\dots$ with $a_i\in\{A, B, C, D\}$ such that $(x, y)\in a_0$, $T(x, y)\in a_1,\dots, T^n(x, y)\in a_n,\dots$, and conversely any compatible sequence $a_0a_1\dots$ defines a unique point $(x, y)\in X$. We describe now the invariant measure we select. $\forall I\in[0, 1]\times[0, 1]$ we set $\mu(T_0^{-1}I)=\frac{1}{2}\mu(I)$, $\mu(T_1^{-1}I)=\frac{1}{2}\mu(I)$. Once these invariance formulas and Markov compatibility rules are stated, the measure of all "cylinders" $a_0\cdots a_n$ can be computed, and μ is the maximum entropy (log 2) Markov invariant measure.

(b) Projection Rules. It is possible, and also easier, to understand the distribution of points $(1-\gamma)\sum\varepsilon_n\gamma^n$ on the line by looking at the two-dimensional system and not just its projection. We consider the γ-adic expansion of $x\in[0, 1]$, $x=\sum_{i>0}\varepsilon_i\gamma^i$, $\varepsilon_i\in\{0, 1\}$. Since $\gamma+\gamma^2=1$, the admissible γ-expansions of x are the sequences $\boldsymbol{\varepsilon}(x)=\varepsilon_1\varepsilon_2\dots$ of 0 and 1 without two adjacent ones.[25] Of course the Markov partition is not necessary for the understanding of the two-dimensional dynamics; it was introduced to set down a "dictionary" for projecting it on the line and vice versa, i.e., a map Φ from the space of the admissible sequences $a_0a_1\dots$ to the space of the admissible sequences $\varepsilon_1\varepsilon_2\dots$: $\Phi(a_0a_1\dots)=\varepsilon_1\varepsilon_2\dots$ We summarize it in Table I.

Note that the shift does not commute with the projection. We were able nevertheless to use these rules to count how many and which Markov sequences have the same given projection.

(c) The Measure ν_γ. We have constructed a map Φ from the space of the admissible sequences $a_0a_1\dots$ to the space of the admissible

Table I[a]

$\mathbf{a}(x,y)$	$\boldsymbol{\varepsilon}(x)$	$\mathbf{a}(T^*(x,y))$	$\boldsymbol{\varepsilon}(T^*_x(x,y))$
$C^C_D a_2 \ldots$	$OO\varepsilon_3 \ldots$	$a_2 a_3 \ldots$	$\varepsilon_3 \varepsilon_4 \ldots$
$CAa_2 \ldots$	$OO\varepsilon_3 \ldots$	$Ca_3 a_4 \ldots$	$1O\varepsilon_5 \ldots$
$DB^A_B a_3 \ldots$	$O1O\varepsilon_4 \ldots$	$a_3 a_4 \ldots$	$\varepsilon_4 \varepsilon_5 \ldots$
$DBDa_3 \ldots$	$O1O\varepsilon_4 \ldots$	$Ba_4 a_5 \ldots$	$OO\varepsilon_6 \ldots$
$AC^C_D a_3 \ldots$	$O1O\varepsilon_4 \ldots$	$a_3 a_4 \ldots$	$\varepsilon_4 \varepsilon_5 \ldots$
$ACAa_3 \ldots$	$O1O\varepsilon_4 \ldots$	$Ca_4 a_5 \ldots$	$1O\varepsilon_6 \ldots$
$B^A_B a_2 \ldots$	$1O\varepsilon_3 \ldots$	$a_2 a_3 \ldots$	$\varepsilon_3 \varepsilon_4 \ldots$
$BDa_2 \ldots$	$1O\varepsilon_3 \ldots$	$Ba_3 \ldots$	$O\varepsilon_4 \ldots$

[a] An asterisk denotes the second or third iterate of T.

sequences $\varepsilon_1 \varepsilon_2 \ldots$: $\Phi(a_0 a_1 \ldots) = \varepsilon_1 \varepsilon_2 \ldots$ We define the projected measure ν_γ as the image of μ via Φ: $\forall$ cylinder $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N$:

$$\nu_\gamma(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N) = \mu(\Phi^{-1}(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N))$$

We call $\#\{\Phi^{-1}(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N)\}$ the "ambiguity" of $a_0 a_1 \ldots a_N$.

(d) The Ambiguity of the Projection. Our aim is "to count ambiguity": the projection rules have been constructed to know which are the Markov rectangles all projecting on the same interval of $[0, 1]$. We concentrate on the central, overlapping zone, whose Markov code is $AC\ldots$ or $DB\ldots$. Observe that the γ-coding of an interval I which lies in "the center"—i.e., is contained in $[\gamma^2, \gamma]$—has the form $\varepsilon(I) = 01$, n_1, 01, $n_2, \ldots$, $n_i \in N$. This repeated structure allows us to use the projection rules just as if there were commutation between projecting and shift. We have to count how many sequences are produced between two consecutive passages through the center, i.e., passages above 01: it is clear that we can express how ambiguity propagates passage after passage by means of products of matrices. These matrices (indexed by n) simply count how many (words terminating with) AC and DB are produced by (a word beginning with) AC in a passage for the center after the time n, and how many AC, DB are produced by DB.

Let

$$B(k) = \begin{pmatrix} 1 & k \\ 1 & k+1 \end{pmatrix}, \qquad A(k) = \begin{pmatrix} 1 & k+1 \\ 1 & k+1 \end{pmatrix}$$

Then we stated that the ambiguity $\#\{\Phi^{-1}(01, n_1 - 1, 01, n_2 - 1, \ldots, n_q - 1)\}$ is given by

$$\left| M(n_q)\, M(n_{q-1}) \ldots M(n_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|$$

where $M(n) = A(k)$ if $n = 2k + 2$ and $M(n) = B(k)$ if $n = 2k + 1$.

Therefore $\{\Phi^{-1}(01,\ n_1-1,\ 01,\ n_2-1,\ 01,\ n_3-1,...\}$ consists of Markov rectangles which are built by connecting the elementary Markov rectangles following the rule that we can connect two of them if and only if the beginning of the following one is equal to the ending of the preceding one.

This means that we consider the Markov system of the space X of the elementary strings $(\cdot, n-1, \cdot)$, $\forall n \geqslant 1$, the Markov measure on strings induced by μ, and the associated transition matrix, denoted Π.

(e) The Lyapunov Exponent.[12,15,16,21] Let (X, μ) be a (discrete) probability space, $\Pi(x, y)$ a Markov transition matrix such that $\mu\Pi = \mu$, and $\Pi^n(i, j) > 0$. Let M: $X \to$ nonnegative matrices of order two, such that $\int \log |M(x)|\ \ d\mu(x) < \infty$. Consider the transition kernel $Q(x, \phi, y, \xi) = \prod(x, y)\ \delta_{M(x)\phi}(\xi)$ on $X \times S^1$ (S^1 is the circle); there is on $X \times S^1$ a measure N left invariant by Q: $NQ = N$; it has the form $N = \mu(x)\ \nu_x(d\phi)$. Consider the ergodic system $(X^N \times S^1,\ \hat{\theta},\ P_\mu \times \nu_{x_0})$, where $\hat{\theta}$:$(x, \phi) \to (\theta\mathbf{x}, M(x_0)\phi)$, where $\{\theta\mathbf{x}\}_n = x_{n+1}$ is the shift on the space X^N of the trajectories $\{x_n\}$ of the Markov process; P_μ is the measure on X^N such that if $x_n(\mathbf{x}) = x_n$, $P_\mu(x_n(\mathbf{x}) = i) = \mu(i)$, and $P_\mu(x_{n+1}(\mathbf{x}) = j\ |x_n(\mathbf{x}) = i) = \prod(j \mid i)$. Let $G(\mathbf{x}, \phi) = \log(|M(x_0)\phi|/|\phi|)$. Then

$$\frac{1}{n}\log\frac{|M(x_n)\ M(x_{n-1})...M(x_0)\phi|}{|\phi|} = \frac{1}{n}\sum_{i=0}^{n-1} G(\hat{\theta}^i(\mathbf{x}, \phi))$$

converges $P_\mu(\mathbf{x}) \times \nu_{x_0}(d\phi)$ almost everywhere to

$$\lambda = \sum_{x_0} \int_{S^1} \log\frac{|M(x_0)\phi|}{|\phi|}\ \mu(x_0)\ \nu_{x_0}(d\phi)$$

Because of the peculiar nature of the family of discrete measure $\{\nu_{x_0}\}$, we were able to write an explicit formula for the exponent:

$$\lambda = \frac{1}{6}\sum_{k\geqslant 0}\ \sum_{\substack{\phi = \binom{h}{h+q},\\ h\geqslant 1, q\geqslant 0}} \Bigg[\log\frac{|A(k)\phi|}{|\phi|}$$

$$\times\left(\frac{1}{2^{2k+2}}\nu_{AC}(\phi) + (k+1)\frac{1}{2^{2k+2}}\left[\nu_{DB}(\phi) + \nu_{BB}(\phi)\right]\right)$$

$$+\log\frac{|B(k)\phi|}{|\phi|}\left(\frac{1}{2^{2k+2}}\nu_{AC}(\phi) + (k+1)\frac{1}{2\cdot 2^{2k+2}}\left[\nu_{DB}(\phi) + \nu_{BB}(\phi)\right]\right)\Bigg]$$

Finally, the dimension of ν_γ is

$$\delta \equiv \dim(\nu_\gamma) = \frac{\lambda - E \log 2}{E \log \gamma}$$

where E log 2 is a normalization and E log γ is the almost sure value of the coding of the length of an interval in $[\gamma^2, \gamma]$.

1.2. Problem II. The Singularity Spectrum of the Golden Mean ICBM

Most of the known multifractal analysis is one-dimensional in essence. All the papers we know present variations and technical improvement over ref. 8, but follow the same general line. Among these, refs. 28 and 30 deal with Axiom A diffeomorphisms and study how one expanding and one contracting direction might interact. Our model is perhaps the first for which it had been possible to obtain a result on multifractal analysis in the case when two different positive rates of expansion interact in a non-trivial way. Although it is a very peculiar model of two-dimensional expansive dynamics, it yields some multifractal analysis for the very classical measure ν_γ, and this is the content of our papers.

Consider the measure ν_γ, the infinitely, convolved Bernoulli measure associated with the golden number $\gamma = (-1 + \sqrt{5})/2$. We know[22] that ν_γ almost surely

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \nu_\gamma(I(x))}{\log |I(x)|} = \delta \tag{1}$$

We plan to study the local exponent:

$$\alpha(x) = \lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \nu_\gamma(I(x))}{\log |I(x)|}$$

if the limit exists. Let $B_0 = \{x : \alpha(x) = \alpha\}$ and let $f(\alpha)$ be the Hausdorff dimension of B_α. Multifractal analysis is concerned with the study of $\{(\alpha, f(\alpha))\}$, the "dimension spectrum" of the measure ν_γ.

Thermodynamic formalism[6,20] provides a by now "classical" method[8] to compute $f(\alpha)$. Let $Z_n = \sum_{I \in A_n} \nu_\gamma(I)^\beta$, where A_n is an exponentially fast (with n) decreasing partition of the system, and let us suppose that the thermodynamic limit $\lim_{n \to \infty} (1/n) \log Z_n$ exist and define a regular function ("pressure") $F(\beta)$. Then, if we denote $\tilde{f}(\alpha)$ the Legendre transform of $F(\beta)$, that is, $\inf_\beta(\alpha\beta - F(\beta))$, then the large-deviation theorem states that $\#\{I : \nu_\gamma(I) \sim |I|^\alpha\}$ behaves as $\exp(n\tilde{f}(\alpha))$ for large n. This result would allow us to show that actually $\tilde{f}(\alpha) = f(\alpha)$, that is, $\tilde{f}(\alpha)$ is the Hausdorff dimension of B_α, the set where the measure has a power-law singularity of strength α. This gives the meaning to $\tilde{f}(\alpha)$ in terms of ν_γ, and moreover provides a method to compute $f(\alpha)$ as Legendre transform of $F(\beta)$. Our model does not allow us to work out estimates on the measure of uniform atoms and therefore we choose to consider a joint partition function $G_n(\beta, F) = \sum_{I \in A_n} \nu_\gamma(I)^\beta \, l(I)^F$, the thermodynamic limit of which can also be studied via the large-deviation theorem. We have to deal with a two-dimensional version of it, because of the joint fluctuations of masses and volumes. Consequently, the dimension of the set of trajectories where the measure has a singularity of strength α will be the Legendre transform $f(\alpha)$ of the (unique) function F realizing the "good" (mass/volume) section of the two-dimensional problem.

Note that $f(\alpha)$, while obtained as a section of a joint large-deviation function $f(\alpha, l)$, is intrinsic to the dynamical system (Ω, f, μ). Indeed, if the pointwise limit

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \mu_\gamma(I(x))}{\log |I(x)|} = \alpha$$

exists and is equal to α on a set B_0 of points x, then the limit exists and is the same for all subsequences of $I(x)$, $x \in B_\alpha$, whose diameter goes to zero. We can then associate to B_α (via the construction of a Frostmann measure related to G) its Hausdorff dimension $f(\alpha)$.

We report in this Part I how the perturbative approach usual in large-deviation theory for random products of matrices[26,4,3] can be applied to our model. For pedagogical reasons, and in order to simplify our formulas, we first set this theory in a simpler context and then we extend it. We state some properties of the thermodynamic function F. Our main result is the strict convexity of the pressure F in a neighborhood of zero, that is, the value of β corresponding to the almost sure value of α. These results will be applied to the multifractal analysis in Part II. The perturbative approach only concludes near zero.

2. CONTRACTION PROPERTIES

In ref. 22 we studied the relations between a Markov partition P_0 for F and the γ-adic partition of the x axis, to establish a dimension formula for the measure ν_γ. The ν_γ measure of a γ-adic interval is computed by counting the rectangles of the Markov partition which project on it. The dimension of the measure is therefore associated with the growth of a random products of Markov matrices. These matrices are

$$M(n) = \begin{cases} \begin{pmatrix} 1 & k+1 \\ 1 & k+1 \end{pmatrix} & \text{if } n = 2k+2 \\ \begin{pmatrix} 1 & k \\ 1 & k+1 \end{pmatrix} & \text{if } n = 2k+1 \end{cases}$$

and $M(n) \equiv M(x_n)$, where $x_n \in F^{-n} P_0$.

The space X^N of $\{x_n\}_{n \in N}$ is a Markov process with distribution P_μ such that if $x_n(\mathbf{x}) = x_n$, then $P_\mu[x_n = i] = \mu(i)$ and $P_\mu[x_{n+1} = j \mid x_n = i] = \pi(j \mid i)$, where the initial distribution μ and the transition matrix $\pi(i \mid j)_{i \in N, j \in N}$ are described in ref. 22. If $n_1 \ldots n_q$ is the coding of a γ-adic interval, then its ν_γ-measure equals

$$\frac{|M(x_{n_q} \ldots M(x_{n_1})|}{2^{(x_{n_1} + \ldots + x_{n_q})}}$$

and its length equals $l(x_{n_1}) + \ldots + l(x_{n_q})$, where $l(x_{n_i})$ is the length of an interval $\in [0, 1]$ whose γ code is 0100...—n_i times 0—001 (see ref. 22 for more details).

The aim of this section is to prove a contraction property of the random matrices $S_n = M(x_n)\, M(x_{n-1}) \ldots M(x_0)$.

If x and y are two vectors of R^2,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we define the distance

$$\delta(\bar{x}, \bar{y}) = \frac{|x_1 y_2 - x_2 y_1|}{\|x\| \cdot \|y\|}$$

Therefore δ is the absolute value of the sinus of the angle between $\bar{x}$ and $\bar{y}$. We denote by S the unit circle in R^2. Let

$$c(M) = \sup_{x, y \in S} \frac{\delta(Mx, My)}{\delta(x, y)}$$

Then

$$c(M) \begin{cases} 1 & \text{if } M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ < 1/5 & \text{if } M = \begin{pmatrix} 1 & k \\ 1 & k+1 \end{pmatrix}, \quad k \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This proves the following result.

Lemma 2.1. The following condition holds:

$$E(c(M)) < \rho < 1$$

and we can take $\rho = 1/2$.

We need also the following result.

Lemma 2.2. The transition matrix $\Pi(j \mid i)$ satisfies the Doeblin condition.

Proof. By ref. 9 it suffices that $S_n(i) = \sum_j^n \Pi(j \mid i)$ converge to 1 uniformly in i. But this is true, because the transition matrix is made up of three repeated rows. Equivalently it can be shown that there exists a positive integer n and an uniformly positive column for the matrix $\Pi^n(j \mid i)$. This can be expressed by the equivalent condition: $\exists n$ such that $\sup_{i,k} \sum_j |p_{ij}^n - p_{kj}^n| < 1$. The Doeblin condition guarantees the existence, uniqueness, and ergodicity of the invariant (under Π) measure μ. Nevertheless, we already know, by construction, that there exists a stationary ergodic measure.

We turn now to the contraction property (with respect to the distance δ) of our matrices. The "singularity" coming from "even" matrices

$$\begin{pmatrix} 1 & k+1 \\ 1 & k+1 \end{pmatrix}$$

for which $c(M) = 0$ can be easily removed by "forgetting" the contribution due to them.

Lemma 2.3. There exist $\rho_1 < 1$ and $c > 0$ such that, uniformly in x_0, u, v and $\forall \eta > 0$

$$E_{x_0}\left(\frac{\delta(S_n u, S_n v)}{\delta(u, v)}\right)^\eta \equiv E_{x_0}\left(\frac{\delta(M(x_n)\dots M(x_0)u, M(x_n)\dots M(x_0)v)}{\delta(u, v)}\right)^\eta \leqslant c\rho_1^n$$

Proof. If $M(x_i)$ is an even matrix, then $\delta(S_n u, S_n v)$ equals zero, so, $\forall n$

$$\begin{aligned} E_{x_0}&\left(\frac{\delta(S_n u, S_n v)}{\cdot\ \delta(u, v)}\right)^\eta \\ &= \sum_{x_1 \dots x_n} \pi(x_1 \mid x_0) \dots \pi(x_n \mid x_{n-1}) \left(\frac{\delta(S_n u, S_n v)}{\delta(u, v)}\right)^\eta \\ &\leqslant \sum_{x_1 \dots x_n \text{ odd}} \pi(x_1 \mid x_0) \dots \pi(x_n \mid x_{n-1}) \left(\frac{\delta(S_n u, S_n v)}{\delta(u, v)}\right)^\eta \end{aligned}$$

Now, for odd matrices, if

$$M \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$\left(\frac{\delta(Mu, Mv)}{\delta(u, v)}\right) \leqslant c(M) < \rho < 1$$

and $c(M) = 1$ if

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Then

$$E_{x_0}\left(\frac{\delta(S_n u, S_n v)}{\delta(u, v)}\right)^{\eta} \leqslant P_{x_0}\{x_1 = \bar{x}_1 \ldots x_n = \bar{x}_n, \bar{x}_i \text{ odd}\} \leqslant c\rho_1^{\eta}$$

We choose in the following $\eta < 1$.

3. THE SPACE $L_{\eta\theta}$ AND THE OPERATOR $T(\beta)$

We introduce the functional space $L_{\eta\theta}$ and the operator $T(\beta)$ on $L_{\eta\theta}$ so that the study of the iterates $T^n(\beta)$ becomes possible: the properties of $T^n(\beta)$ allow us to infer easily the limit behavior of the partition function G_n (see Section 1).

In this section we state some properties of quasicompactness of $T(0)$. Then we use the perturbation theory of quasicompact operators.

Definition 3.1. Let $S_{[\pi/4, \pi/2]}$ be the circle sector $\{u \in S, \pi/4 < u < \pi/2\}$. Let us fix θ positive and $0 < \eta < 1$. Let $L_{\eta\theta}$ be the space of the functions ϕ: $X \times S_{[\pi/4, \pi/2]} \to C$ such that

$$\|\phi(x, u)\|_{\eta, \theta} = \sup_{x_0, u} \frac{|\phi(x_0, u)|}{(\|x_0\| + 1)^{\theta}} + \sup_{x_0, u \neq v} \frac{|\phi(x_0, u) - \phi(x_0, v)|}{(\|x_0\| + 1)^{\theta}\, \delta(u, v)^{\eta}} < \infty$$

We write

$$\|f(x_0, u)\|_{\eta, \theta} = f_{\infty} + m_{\eta}(f)$$

Let us introduce the operators on $L_{\eta,\theta}$:

$$T(\beta)f(x_0, u) = E_{x_0}(e^{\beta \log |M(x_0)u|/|u|} f(x_1, M(x_0 u)))$$

$$T(0)f(x_0, u) = Pf(x_0, u) = \sum_{x_1} \pi(x_1 \mid x_0) f(x_1, M(x_0 u))$$

$$Nf(x_0, u) = \sum_{x_0} \sum_{V} \pi(x_0)\, \nu_{x_0}(v) f(x_0, v)$$

We state the following result.

Proposition 3.2. P is a bounded operator on $L_{\eta\theta}$.

Proof. We have

$$\frac{|Pf(x_0, u)|}{(\|x_0\|+1)^\theta} = E_{x_0}\left(\frac{f(x_1, M(x_0)u)}{(\|x_0\|+1)^\theta}\right)$$

$$\leqslant \sup_{x_0, u} \frac{|f(x_0, u)|}{(\|x_0\|+1)^\theta} = |f|_\infty$$

Also,

$$\frac{|Pf(x_0, u) - Pf(x_0, v)|}{(\|x_0\|+1)^\theta\, \delta(u, v)^\eta}$$

$$\leqslant E_{x_0}\left(\frac{|f(x_1, M(x_0 u)) - f(x_1, M(x_0 v))|}{|\|x_1\|+1)^\theta\, (\delta(M(x_0 u), M(x_0 v)))^\eta}\right.$$

$$\left.\times \frac{(\delta(M(x_0 u), M(x_0 v)))^\eta}{(\delta(u, v))^\eta} \frac{(\|x_1\|+1)^\theta}{(\|x_0\|+1)^\theta}\right)$$

$$\leqslant m_\eta(f)\, E_{x_0}((c(M(x_0)))^\eta\, (\|x_1\|+1)^\theta) \leqslant c m_\eta(f)$$

We show that P is a quasicompact operator:

Proposition 3.3. The following condition holds:

$$\lim_{n \to \infty} \|P^n - N\|_{\eta,\theta}^{1/n} < 1$$

Proof. We have

$$P^n f(x_0, u) = \sum_{x_0 \ldots x_n} \pi(x_1 \mid x_0) \ldots \pi(x_n \mid x_{n-1}) f(x_n, M(x_{n-1}) \ldots M(x_0)u)$$

$$Nf(x_0, u) = \sum_{x_0} \sum_{v} \pi(x_0)\, \nu_{x_0}(v) f(x_0, v)$$

Then

$$\frac{|(P^n-N)f(x_0,u)-(P^n-N)f(x_0,v)|}{(\|x_0\|+1)^\theta\,\delta(u,v)^\eta}$$

$$=\frac{|P^nf(x_0,u)-P^nf(x_0,v)|}{(\|x_0\|+1)^\theta\,\delta(u,v)^\eta}$$

$$\leqslant m_\eta(f)\left[E_{x_0}\left(\frac{(\delta(S_{n-1}u,S_{n-1}v))^{2\eta}}{(\delta(u,v))^{2\eta}}\right)\right]^{1/2}[E_{x_0}((\|x_n\|+1)^{2\theta}]^{1/2}$$

$$\leqslant m_\eta(f)\left[\sum_{x_0\ldots x_n\,\mathrm{odd}}\left(\frac{(\delta(S_{n-1}u,S_{n-1}v))^{2\eta}}{(\delta(S_{n-2}u,S_{n-2}v))^{2\eta}}\cdots\frac{(\delta(Su,Sv))^{2\eta}}{(\delta(u,v))^{2\eta}}\right)\right]^{1/2}\left(\sum_k\frac{k^{2\theta}}{2^k}\right)^{1/2}$$

—as in Lemma 2.3:

$$\leqslant m_\eta(f)\,cp^{n/2}\leqslant m_\eta(f)\,cp_2^{n/2}$$

Similarly,

$$\frac{|(P^n-N)f(x_0,u)|}{(\|x_0\|+1)^\theta}=E_{x_0}\left(f(x_n,M_{x_{n-1}}\ldots M_{x_0}u)-\sum_{x_0}\sum_v\pi_{x_0}\nu_{x_0}(v)f(x_0,v)\right)$$
$$\times(\|x_0\|+1)^{-\theta}$$

Note that $\pi\times\nu$ is invariant under $P=T(0)$, so

$$\sum_{x_0}\sum_v\pi(x_0)\,\nu_{x_0}(v)f(x_0,v)=\sum_{x_0}\sum_u\pi_{x_0}\nu_{x_0}(u)\,T^n(0)f(x_0,u)$$

Then

$$E_{x_0}\left(f(x_n,S_{n-1}u)-\sum_{x_0}\pi(x_0)\sum_v\nu_{x_0}(v)f(x_n,S_{n-1}v)\right)\cdot(\|x_0\|+1)^{-\theta}$$

$$=E_{x_0}\left(\sum_{x_0}\pi(x_0)\sum_v\nu_{x_0}(v)(f(x_n,S_{n-1}u)-f(x_n,S_{n-1}u))\right)\cdot(\|x_0\|+1)^{-\theta}$$

If one of the matrices of the product S_n is even, $\forall u,v$, then $S_nu=S_nv$ and the above sum equals zero. Then, we rewrite this sum as

$$E_{x_0}\left(\sum_{x_0,v}\pi(x_0)\,\nu_{x_0}(v)\frac{|f(x_n,S_{n-1}u)-f(x_n,S_{n-1}v)|}{(\delta(S_{n-1}u,S_{n-1}v))^\eta}\right.$$
$$\left.\times\frac{(\delta(S_{n-1}u,S_{n-1}v))^\eta}{(\delta(u,v))^\eta}\frac{(\|x_n\|+1)^\theta}{(\|x_0\|+1)^\theta(\|x_n\|+1)^\theta}\right)$$

$$\leqslant m_\eta(f) \sum_{x_0, v} \pi(x_0)\, v_{x_0}(v)$$

$$\times \left[\sum_{x_1 \dots x_n \text{ odd}} \pi(x_n \mid x_{n-1}) \dots \pi(x_1 \mid x_0) \frac{(\delta(S_{n-1}u, S_{n-1}v))^{2\eta}}{(\delta(u, v))^{2\eta}} \right]^{1/2}$$

$$\times \left[\sum_{x_1 \dots x_n} \pi(x_n \mid x_{n-1}) \dots \pi(x_1 \mid x_0) \frac{(\|x_n + 1)^{2\theta}}{(\|x_0\| + 1)^{2\theta}} \right]^{1/2}$$

$$\leqslant m_\eta(f) \sum_{x_0, v} \pi(x_0)\, v_{x_0}(v)$$

$$\times \left[\sum_{x_1 \dots x_n \text{ odd}} \pi(x_n \mid x_{n-1}) \dots \pi(x_1 \mid x_0) \right.$$

$$\left. \times \frac{(\delta(S_{n-1}u, S_{n-1}v))^{2\eta}}{(\delta(S_{n-2}u, S_{n-2}v))^{2\eta}} \cdots \frac{(\delta(Mu, Mv))^{2\eta}}{(\delta(u, v))^{2\eta}} \right]^{1/2}$$

$$\times \left[\sum_{x_1 \dots x_n} \pi(x_n \mid x_{n-1}) \dots \pi(x_1 \mid x_0) \frac{(\|x_n\| + 1)^{2\theta}}{(\|x_0\| + 1)^{2\theta}} \right]^{1/2}$$

$$\leqslant m_\eta(f) \sum_{x_0} \pi(x_0) \sum_{v} v_{x_0}(v)$$

$$\times \left[\sum_{x_1 \text{ odd}} \pi(x_1 \mid x_0)\, \rho^n \right]^{1/2} \left(\sum_k \frac{k^{2\theta}}{2^k} \right)^{1/2} \leqslant m_\eta(f)\, \bar{c} \rho^n$$

In conclusion, we have shown that

$$\|P^n - N\|_{\eta,\theta} = \|(P^n - N) f\|_\infty + m_\eta((P^n - N) f)$$
$$\leqslant c\rho^n (m_\eta(f) + |f|_\infty)$$

which implies that

$$\frac{\|(P^n - N) f\|_{\eta,\theta}}{\|f\|_{\eta,\theta}} \leqslant \rho^n$$

As $(P^n - N) = (P - N)^n$, then the limit $\lim_{n \to \infty} \|P^n - N\|_{\eta,\theta}^{1/n}$ equals the limit $\lim_{n \to \infty} \|(P - N)^n\|_{\eta,\theta}^{1/n}$. This limit exists by subadditivity and is smaller than 1 by the above estimates.

4. PERTURBATIONS OF THE OPERATOR T

In this section we set some regularity properties of the family of operators $T(\beta)$.

Let β be complex, $|\beta|$ small. Following ref. 3, we introduce the operator

$$T(\beta) f(x_0, u) = E_{x_0}(e^{\beta \log[|M(x_0)u|/|u|]} f(x_1, M(x_0 u)))$$

whose nth iterate is

$$T^n(\beta) f(x_0, u) = E_{x_0}(e^{\beta \log[|M(x_n)\dots M(x_0)u|/|u|} f(x_n, M(x_{n-1}\dots M(x_0 u))))$$

because of the cocycle property of $[\log |M(x_0)u|/|u|$.

Following ref. 3, we have the following result.

Lemma 4.1. Let $\operatorname{Re}\beta < \theta$. There exists three positive constants c_1, c_2, c_3 such that:

$$\text{(a)} \qquad E_{x_0}\left(\frac{e^{\beta \log |M(x_0)u|/|u|} - e^{\beta \log |M(x_0)v|/|v|}}{(\delta(u, v))^\eta \, (\|x_0\| + 1)^\theta}\right) \leqslant c_1$$

$$\text{(b)} \qquad E_{x_0}\left(\frac{\log |M(x_0)u|/|u| - \log |M(x_0)v|/|v|}{(\delta(u, v))^\eta \, (\|x_0\| + 1)^\theta}\right) \leqslant c_2$$

$$\text{(c)} \qquad E_{x_0}\left(\frac{e^{\beta \log |M(x_0)u|/|u|}}{(\|x_0\| + 1)^\theta}\right) \leqslant c_3$$

Proof of (a). We have

$$\begin{aligned} \|Mu\|^\beta - \|Mv\|^\beta &\leqslant |\beta| \frac{\|M\|}{\|Mu\|^{1-\beta}} \|u - v\| \\ &\leqslant |\beta| \cdot \|M\| \cdot \|M\|^{\operatorname{Re}\beta - 1} \|u - v\| \end{aligned}$$

so that the expression in (a) can be bounded by

$$\text{(a)} \leqslant \sup_{u,v} E_{x_0}\left(\frac{\|M\|^{\operatorname{Re}\beta} \|u - v\|}{(\delta(u, v))^\eta \, (\|x_0\| + 1)^\theta}\right)$$

Since $\|u - v\| \leqslant \sqrt{2}\, \delta(u, v) \leqslant \sqrt{2}\, \delta(u, v)^\eta$, we have

$$\text{(a)} \leqslant \sqrt{2}\, E_{x_0}\left(\frac{\|M(x_0)\|^{\operatorname{Re}\beta}}{(\|x_0\| + 1)^\theta}\right) \leqslant \sqrt{2}$$

Proof of (b). We have

$$\log \|Mu\| - \log \|Mv\| \leqslant \sup_u \frac{\|M\|}{\|Mu\|} \|u - v\|$$

so that

$$(\text{b}) \leqslant \sup_{u,v} E_{x_0}\left(\frac{(\|M\|/\|Mu\|)\ \|u-v\|}{(\delta(u,v))^{\eta}\ (\|x_0\|+1)^{\theta}}\right)$$

$$\leqslant \sup_{u,v} E_{x_0}\left(\frac{\|M\|}{\|Mu\|}\frac{\delta(u,v)}{(\delta(u,v))^{\eta}}\frac{1}{(\|x_0\|+1)^{\theta}}\right) \leqslant c_2$$

Proof of (c). We have

$$(\text{c}) = E_{x_0}\left(\frac{\|M(x_0)\|^{\beta}}{(\|x_0\|+1)^{\theta}}\right) \leqslant E_{x_0}\left(\frac{(\|x_0\|+1)^{\operatorname{Re}\beta}}{(\|x_0\|+1)^{\theta}}\right) \leqslant c_3$$

Proposition 4.2. The family $\{T(\beta)\}_{\beta, \operatorname{Re}\beta < \theta/2}\}$ is a family of class C^k of bounded operators on $L_{\eta,\theta}$.

Proof. We have

$$\frac{|T(\beta)f(x_0,u)|}{(\|x_0\|+1)^{\theta}} = E_{x_0}\left(\frac{e^{\beta\log[|M(x_0)u|/|u|]}f(x_1, M(x_0u))}{(\|x_0\|+1)^{\theta}}\right)$$

$$\leqslant |f|_{\infty}\, E_{x_0}(\|x_1\|+1)^{\theta}\frac{|M(x_0)u|^{\beta}}{(\|x_0\|+1)^{\theta}} \leqslant c\ |f|_{\infty}$$

if $\theta \geqslant \operatorname{Re}\beta$.

Similarly,

$$\frac{|T(\beta)f(x_0,u) - T(\beta)f(x_0,v)|}{(\|x_0\|+1)^{\theta}\,(\delta(u,v))^{\eta}}$$

$$\leqslant E_{x_0}\left(e^{\beta[\log|M(x_0)u|]/|u|}\frac{f(x_1, M(x_0u)) - f(x_1, M(x_0v))|}{(\|x_1\|+1)^{\theta}\,(\delta(Mu, Mv))^{\eta}}\right.$$

$$\left.\times\frac{(\delta(Mu,Mv))^{\eta}}{(\delta(u,v))^{\eta}}\frac{|\|x_1\|+1)^{\theta}}{(\|x_0\|+1)^{\theta}}\right)$$

$$+E_{x_0}\left(\frac{e^{\beta\log|M(x_0)u|} - e^{\beta\log|M(x_0)v|}}{(\delta(u,v))^{\eta}}\frac{f(x_1, M(x_0u))(\|x_1\|+1)^{\theta}}{(\|x_1\|+1)^{\theta}\,(\|x_0\|+1)^{\theta}}\right)$$

$$\leqslant m_{\eta}(f)\sup_{u,v} E_{x_0}\left(\frac{|M(x_0)u|^{\beta}}{(\|x_0\|+1)^{\theta}}c(M(x_0))^{\eta}\,(\|x_0\|+1)^{\theta}\right)$$

$$+c_1\,|f|_{\infty}\sup_{u,v} E_{x_0}\left(\|M(x_0)\|^{\operatorname{Re}\beta}\frac{\delta(u,v)}{(\delta(u,v))^{\eta}}\frac{(\|x_1\|+1)^{\theta}}{(\|x_0\|+1)^{\theta}}\right)$$

by Lemma 4.1. Again by Lemma 4.1,

$$\leqslant m_{\eta}(f)\, c_4 + c_5\,|f|_{\infty}$$

To prove differentiability, write

$$\frac{d^n}{d\beta^n} T(\beta) f(x_0, u) = E_{x_0}(\log^n |M(x_0)u| \, e^{\beta \log[|M(x_0)u|/|u|]} f(x_1, M(x_0 u)))$$

and compute the $L_{\eta,\theta}$-norm of the derivatives:

$$\left\| \frac{d^n}{d\beta^n} T(\beta) f \right\|_{\eta,\theta} = \left| \frac{d^n}{d\beta^n} T(\beta) f \right|_{\infty} + m_\eta \left(\frac{d^n}{d\beta^n} T(\beta) f \right)$$

The second term is bounded as follows:

$$\frac{|(d/d\beta^n)\, T(\beta) f(x_0, u) - (d/d\beta^n)\, T(\beta) f(x_0, v)|}{(\|x_0\| + 1)^\theta \, (\delta(u, v))^\eta}$$

$$\leqslant m_\eta(f)\, E_{x_0} \left(\frac{|M(x_0)u|^\beta \log^n |M(x_0)u|}{(\|x_0\| + 1)^\theta} \cdot (c(M_{x_0}))^\eta \, (\|x_1\| + 1)^\theta \right)$$

$$+ |f|_\infty \, E_{x_0} \left(\frac{|M(x_0 u|^\beta \, |\log |M(x_0)u| - \log |M(x_0)v| \,|}{(\delta(u, v))^\eta \, (\|x_0\| + 1)^\theta} (\|x_1\| + 1)^\theta \right.$$

$$\left. \times \sum_{i=0}^{n-1} \log |Mu|^i \log |Mu|^{n-i-1} \right)$$

$$+ |f|_\infty \, E_{x_0} \left(\frac{\log^n |M(x_0)u| \, (|\,|M(x_0)u|^\beta - \log |M(x_0)v|^\beta|)}{(\delta(u, v))^\eta \, (\|x_0\| + 1)^\theta} \right.$$

$$\left. \times (\|x_1\| + 1)^\theta \right)$$

[because $a^n - b^n = (a - b) \sum_0^{n-1} a^i b^{n-i-1}$]

$$\leqslant m_\eta(f) \left[E_{x_0} \left(|M(x_0)u|^{2\beta} \frac{\log^{2n} |M(x_0)u|}{(\|x_0\| + 1)^{2\theta}} \right) \right]^{1/2}$$

$$\times [E_{x_0}((\|x_1\| + 1)^{2\theta}]^{1/2}$$

$$+ |f|_\infty \left[E_{x_0} \left(\frac{|\log |M(x_0)u| - \log |M(x_0)v| \,|^2}{(\delta(u, v))^{2\eta} \, (\|x_0\| + 1)^{2\theta}} \right) \right]^{1/2}$$

$$\times \left[E_{x_0} \left(\frac{|M(x_0)u|^{2\beta} \, n^2 \, |\log^{2n} |M(x_0)| \,|}{(\|x_0\| + 1)^{2\theta}} (\|x_1\| + 1)^{2\theta} \right) \right]^{1/2}$$

$$+ |f|_\infty \left[E_{x_0} \left(\frac{\log^{2n} |M(x_0)u| \, (\|x_1\| + 1)^{2\theta}}{(\|x_0\| + 1)^\theta} \right) \right]^{1/2}$$

$$\times \left[E_{x_0} \left(\frac{|\,|M(x_0)u|^\beta - |M(x_0)v|^\beta|}{(\|x_0\| + 1)^{\theta/2} \, (\delta(u, v))^\eta} \right)^2 \right]^{1/2}$$

which, by Lemma 4.1, is bounded by $C(m_\eta(f) + |f|_\infty)$, where C is a constant.

Similarly, for the first term

$$\frac{|(d^n/d\beta^n)\, T(\beta) f(x_0, u)|}{(\|x_0\|+1)^\theta}$$

$$= E_{x_0}\left(\frac{|M(x_0)u|^\beta \log^n |M(x_0)u| f(x_1, M(x_0 u))}{(\|x_1\|+1)^\theta} \frac{(\|x_1\|+1)^\theta}{(\|x_0\|+1)^\theta}\right)$$

$$\leqslant C\,|f|_\infty$$

5. SPECTRAL PROPERTIES

We use the C^k-perturbation theory of operators to deduce the spectral properties of $T(\beta)$. Following refs. 3 and 4, we state the following results.

Proposition 5.1. If $T(\beta)$ is a C^k family of operators on $L_{\eta,\theta}$ and if

$$\rho = \lim_{n\to\infty} \|T^n(0) - N\|_{\eta\theta}^{(1/n)} < 1$$

where N is a rank-1 operator on $L_{\eta\theta}$, then, in a neighborhood of 0,

$$T(\beta) = \lambda(\beta)\, N(\beta) + Q(\beta)$$

where (1) $\lambda(\beta)$ is the simple maximal isolated eigenvalue of $T(\beta)$ and $\lambda(0)=1$; (2) $N(\beta)$ is the rank-1 projector associated with λ and $N(\beta)\, Q(\beta) = Q(\beta)\, N(\beta) = 0$; (3) $\beta \to \lambda(\beta)$, $\beta \to N(\beta)$, $\beta \to Q(\beta)$ are functions of class C^k; and (4) $|\lambda(\beta)| > (2+\rho)/3$ and $\forall p \leqslant k$

$$\left|\frac{d}{d^p\beta}\, Q^n(\beta)\right| \leqslant \left(\frac{1+2\rho}{3}\right)^n$$

This theorem prepares the way for the large-deviation results we shall establish in the next section.

6. LARGE DEVIATIONS OF THE AMBIGUITY

Here we obtain a large-deviation theorem as a corollary of the preceding theory. We have first a corollary.

Corollary 6.1. We have

$$\lambda'(0) = \lambda$$

By Proposition 5.1, for all (x_0, u),

$$\lambda'(0) = \lim \frac{1}{n} \frac{d}{d\beta} T^n 1(x_0, u)|_{\beta=0}$$

We have

$$\frac{d}{d\beta} T^n 1(x_0, u)|_{\beta=0} = E_{x_0} \log |M(x_n) \dots M(x_0)u|$$

so that

$$\lambda'(0) = \lim_{n \to \infty} \frac{1}{n} E_{x_0} \log |M(x_n) \dots M(x_0)u|$$

which equals λ by the ergodic theorem.

For $|\beta|$ sufficiently small, set

$$Z_n(\beta)(x_0, u) = E_{x_0} e^{\beta \log |M(x_n) \dots M(x_0)u|} e^{-n\lambda\beta}$$

and

$$F_1(\beta) \equiv \log \lambda(\beta) - \lambda\beta$$

Corollary 6.2. For any (x_0, u)

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta)(x_0, u) = F_1(\beta)$$

the function F_1 is convex, $F_1(0) = 0$, and $dF_1/d\beta = \lambda'(\beta)/\lambda(\beta) - \lambda$; in particular, $F_1'(0) = 0$.

Definition 6.3. We have

$$f_1(\alpha) = \inf_{|\beta| < \bar{\beta}} \{F_1(\beta) - \alpha\beta\}$$

Definition 6.4. We define

$$\sigma^2 \equiv \lambda''(0) - \lambda'^2(0)$$

We can finally state the large-deviation theorem (see, e.g., ref. 27):

Proposition 6.5. If σ^2 is strictly positive, then $F_1(\beta)$ is strictly convex in a neighborhood of zero $[F_1(\beta) \sim \sigma^2\beta^2]$ and there exists τ such that the function $f_1(\alpha)$ is defined for $|\alpha| < \tau$, is strictly convex $[f_1(\alpha) \sim -(\alpha^2/\sigma^2)]$, and

$$\frac{1}{n}\log P_{x_0}(\log |M(x_n)\ldots M(x_0)u| - n\lambda > n\alpha) \xrightarrow[n\to\infty]{} f_1(\alpha) \quad \text{if} \quad -\tau < \alpha < 0$$

$$\frac{1}{n}\log P_{x_0}(\log |M(x_n)\ldots M(x_0)u| - n\lambda < n\alpha) \xrightarrow[n\to\infty]{} f_1(\alpha) \quad \text{if} \quad 0 < \alpha < \tau$$

Remark. By the spectral theorem,

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\frac{d}{d\beta^2} Z_n(\beta)|_{\beta=0}$$

i.e.,

$$\sigma^2 = \lim \frac{1}{n} E_{x_0}(\log |M(x_n)\ldots M(x_0)u| - n\lambda)^2$$

$$= \lim E_{x_0}\left(\frac{S_n}{\sqrt{n}} - \sqrt{n}\,\lambda\right)^2$$

where

$$S_n = \log |M(x_0)u + \log \frac{|M(x_1)\, M(x_0)u}{M(x_0)u} + \ldots \log \frac{|M(x_n)\ldots M(x_0)u|}{M(x_{n-1})\ldots M(x_0)u}$$

σ^2 is therefore the variance of the random variable $S_n/\sqrt{n}$.

7. STRICT POSITIVITY OF THE VARIANCE

The following argument is due to Ph. Bougerol. The same argument allows us to show, in the next section, that the pressure $F(\beta) = \lim_{n\to\infty} (1/n) \log Z_n(\beta)$ is a strictly convex function in a neighborhood of zero.

We can state the following theorem, by refs. 21 and 3.

Proposition 7.1. There exists ϕ in $L_{\eta 0}$ such that

$$\sigma^2 = \sum_{x_0} \sum_{v} \pi(x_0)\, \nu_{x_0}(v) \times \left(\sum_{x_1} \pi(x_1 \mid x_0)(\log |M(x_0)u| - \lambda + \phi(x_1, M(x_0)u) - \phi(x_0, u))^2 \right)$$

and ϕ solves $\phi - T(0)\phi = \psi$, $\psi(x_0, u) = E_{x_0}(\log M(x_0 u)) - \lambda$ ("cocycle property").

Proposition 7.2. The following condition holds:

$$\sigma^2 > 0$$

Proof. We know that

$$\sigma^2 = \sum_{x_0} \sum_{v} \pi(x_0)\, v_{x_0}(v)$$
$$\times \sum_{x_1} (\log |M(x_0)u| - \lambda + \phi(x_1, M(x_0)u) - \phi(x_0, u))^2\, \pi(x_1 \,|x_0)$$

If σ^2 is zero, then for $\pi \times v$ almost all (x_0, u), P_{x_0} almost all x,

$$\log |M(x_0)u| - \lambda + \phi(x_1, M(x_0)u) - \phi(x_0, u) = 0$$

By stationarity of $\pi \times v$ we also have for almost all $(x_0, \ldots, x_n, u)$

$$\phi(x_n, M(x_{n-1}) \ldots M(x_0)u)$$
$$- \sum_{x_{n+1}} \pi(x_{n+1} \mid x_n)\, \phi(x_{n+1}, M(x_n) \ldots M(x_0)u)$$
$$= \log \frac{|M(x_n) \ldots M(x_0)u|}{|M(x_{n-1}) \ldots M(x_0)u|} - \lambda$$

Summing in n yields for almost all $(x_0, \ldots, x_n, u)$

$$\phi(x_n, M(x_{n-1}) \ldots M(x_0)u)$$
$$- \log |M(x_{n-1}) \ldots M(x_0)u| - n\lambda - \phi(x_0, u) = 0$$

We can therefore state the following result.

Proposition 7.3. σ^2 is zero if and only if $\pi \times v$-almost all (x_0, u) and $\forall n$

$$\log |M(x_n) \ldots M(x_0)u| - \lambda(n+1)$$
$$= \phi(x_{n+1}, M(x_n) \ldots M(x_0)u) - \phi(x_0, u) \qquad P^n_{x_0}\text{-a.s.}$$

Let us show that these equalities are impossible. Since v is a discrete measure, it is sufficient to choose a point $\binom{1}{1}$ with $v(\binom{1}{1}) > 0$ and to show that $\log(|M(x_n) \ldots M(x_0)u|\, e^{-(n+1)\lambda}$ is bounded below by λn on a set of positive measure for $P^n_{x_0}$, wile ϕ stays bounded on the same set.

Choose

$$M(x_0) = \ldots = M(x_n) = M(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Then $P^n(x_0=0,\dots x_n=0)\sim 2^{-n}>0$ and

$$\log\left(\left|M(x_n)\dots M(x_0)\begin{pmatrix}1\\1\end{pmatrix}\right|=\log\left|\begin{pmatrix}1&1\\n&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right|=\log(n+3)\right.$$

and $|\log(n+3)\,e^{-(n+1)\lambda}|$ grows as n, whereas

$$|\phi(x_{n+1}, M_{x_n}\dots M_{x_0}u)-\phi(x_0, u)|\leqslant 2\left|\phi\left(0, \begin{pmatrix}2\\n+1\end{pmatrix}\right)\right|$$

and since ϕ is in $L_{\eta,\theta}$ we have that $|\phi(x_0, u)|/(\|x_0\|+1)^\theta\leqslant c$.

So,

$$2\left|\phi\left(0, \begin{pmatrix}2\\n+1\end{pmatrix}\right)\right|\leqslant\frac{c}{(\|0\|+1)^\theta}=\frac{c}{(4+1)^\theta}=C$$

8. THE "NORMALIZED" OPERATOR

This is the operator

$$T(\beta)f(x_0, u)=E_{x_0}(e^{\beta[\log|M(x_0)u|]/|u|}e^{-\beta g(x_0)}f(x_1, M(x_0u)))$$

where $g(x_0)=\log 2^{n+1}$ if $x_0=(\dots n\dots)$.

The nth iterate is

$$\begin{aligned}&T^n(\beta)f(x_0, u)\\&=E_{x_0}(e^{[\beta\log|M(x_n)\dots M(x_0)u]/|u|}e^{-\beta g(x_n)+\dots+g(x_0)}f(x_n, M(x_{n-1}\dots M(x_0u))))\end{aligned}$$

All the properties stated in the preceding sections remain true for the above operator T modulo a slight change of the space $L_{\eta\theta}$. The natural space $L_{\eta\theta}$ is now the space of functions such that

$$\|\cdot\|_{\eta\theta}=\sup_{x_0,u}\frac{|\phi(x_0, u)|}{2^{\theta\|x_0\|}}+\sup_{x_0,u\neq v}\frac{|\phi(x_0, u)-\phi(x_0, v)|}{2^{\theta\|x_0\|}\delta(u, v)^\eta}<\infty$$

Note that the arguments of Section 7 are again true with $|\phi|<2^\theta$ and $|\log Z_n e^{-n\lambda}|\sim n$, and $\theta<1$.

9. THE "JOINT" OPERATOR

For application to the multifractal spectrum the results of Sections 7 and 8 are not sufficient. We introduce the operator "joint partition function" and we modify slightly the functional spaces we work with. We use the preceding theory to prove notably the strict convexity of the pressure in a neighborhood of zero, and to state a large-deviation theorem.

So we finally consider the operator

$$T(\beta_1, \beta_2) f(x_0, u) = E_{x0}(e^{[\beta_1 \log |M(x_0)u|]/|u|} e^{-\beta_1 g(x_0)} e^{\beta_2 l(x_0)} f(x_1, M(x_0 u)))$$

where $g(x_0) = \log 2^{\|x_0\|+1}$ and $l(x_0) = \log \gamma^{\|x_0\|+1}$ if $x_0 = (\ldots n \ldots)$.
The nth iterate is

$$T^n(\beta_1, \beta_2) f(x_0, u) = E_{x0}(e^{[\beta_1 \log |M(x_n)\ldots M(x_0)u|]/|u|} e^{-\beta_1 g(x_n) + \ldots + g(x_0)}$$
$$\times e^{\beta_2 l(x_n) + \ldots + l(x_0)} f(x_n, M(x_{n-1} \ldots M(x_0 u))))$$

Definition 9.1. Let θ_1, θ_2 be two positive real numbers. Let $L_{\eta,\theta_1,\theta_2}$ be the space of functions $f: X \times S \to C$ such that $\|f\|_{\eta,\theta_1,\theta_2} < \infty$, where

$$\|f\|_{\eta,\theta_1,\theta_2} = \sup_{x_0, u} \frac{|\phi(x_0, u)|}{2^{\theta_1 \|x_0\|}(\|x_0\| + 1)^{\theta_2}} + \sup_{x_0, u \neq v} \frac{|\phi(x_0, u) - \phi(x_0, v)|}{2^{\theta \|x_0\|}(\|x_0\| + 1)^{\theta_2} \delta(u, v)^\eta}$$

We have the analog of Proposition 4.2.:

Proposition 9.2. $T(\beta_1, \beta_2)$ is a family of class C^k on $D_1 \times D_2$ of bounded operators of $L_{\eta\theta_1\theta_2}$, where D_i, $i = 1, 2$, is the disk of center 0 and radius $|\beta_i| < \theta_i/2 < 1/2$ in C.

Similarly we have a spectral theorem Analog Proposition to 5.1.

Let us introduce $\forall(x_0, u)$

$$G(\beta_1, \beta_2) = \lim_{n \to \infty} \frac{1}{n} \log T^n(\beta_1, \beta_2)\, l(x_0, u)$$

This limit exists by the spectral theorem.

Proposition 9.3. There exists a function F defined on a neighborhood of zero such that $F(0) = 0$ and $G(\beta, F(\beta)) = 0$. Moreover, $F(\beta)$ is of class C^k.

Proof. By the implicit function theorem, it is sufficient to show that $(\partial/\partial y)\ G(x, y) \neq 0$ for x, y in a neighborhood of zero. By the spectral theorem, we can compute $(\partial/\partial y)\ G$, the derivative of the limit, as the limit of the derivatives, that is, as the limit, for $n \to \infty$, of

$$E_{x0}(e^{[x \log |M(x_n)\ldots M(x_0)u|]/|u|} e^{-xg(x_n) + \ldots + g(x_0)}$$
$$\times e^{yl(x_n) + \ldots + l(x_0)} (l(x_{n-1}) + \ldots + l(x_0)))$$
$$\times \{E_{x0}(e^{[x \log |M(x_n)\ldots M(x_0)u|]/|u|} e^{-xg(x_n) + \ldots + g(x_0)} e^{yl(x_n) + \ldots + l(x_0)})\}^{-1}$$

which we rewrite as

$$\frac{E_{x_0}(e^{x\log S_n}e^{-xg_n}e^{yl_n}(l_n))}{E_{x_0}(e^{x\log S_n}e^{-xg_n}e^{yl_n})}$$

and $(\partial/\partial y)\ G(0,0)=E(x_0)\log\gamma$, which we know to be <0. We also know, by now, that $G(x,y)$ is strictly convex in its two variables separately, by the arguments of Section 7. Then, by continuity, $(\partial/\partial y)\ G<0$ in a neighborhood of zero.

Corollary 9.4. We have

$$\frac{\partial}{\partial\beta}G(\beta,F)+\frac{\partial}{\partial F}G(\beta,F)\cdot\frac{\partial}{\partial\beta}F(\beta)=0$$

Just differentiate the equality $G(\beta,F(\beta))=0$.

Definition 9.5. Let $\alpha(\beta)\equiv-(\partial/\partial\beta)\ F(\beta)$.

Corollary 9.6. We have

$$\alpha(\beta)=\frac{(\partial/\partial\beta)G}{(\partial/\partial F)G}$$

In particular, with the notations of ref. 22,

$$\alpha(0)=\frac{\lambda-E\log 2}{E\log\gamma}=\delta$$

and $0<\delta<1$.

The most important property of F is given in the following result.

Proposition 9.7. For $|\beta|$ sufficiently small

$$\frac{\partial^2}{\partial\beta^2}F>0$$

Proof. Differentiating $G(\beta,F(\beta))=0$, we have

$$\frac{\partial}{\partial\beta}G(\beta,F(\beta))+\frac{\partial}{\partial F}G(\beta,F(\beta))\frac{\partial}{\partial\beta}F(\beta)$$

and

$$\frac{\partial^2}{\partial\beta^2} G(\beta, F) + 2\frac{\partial}{\partial\beta} F \frac{\partial^2}{\partial\beta\,\partial F} G(\beta, F)$$
$$+\frac{\partial^2}{\partial\beta^2} F \frac{\partial}{\partial F} G(\beta, F) + \left(\frac{\partial}{\partial\beta} F\right)^2 \frac{\partial^2}{\partial F^2} G(\beta, F) = 0$$

Let $F(\beta) = -\delta\beta + \sigma_1^2(\beta^2/2) + \dots$ (F being regular, we can use the implicit function theorem to obtain an expansion of F near zero):

$$\frac{\partial^2}{\partial\beta^2} G(0,0) - 2\delta \frac{\partial^2}{\partial\beta\,\partial F} G(0,0) + \sigma_1^2 \frac{\partial}{\partial F} G(0,0) + \delta^2 \frac{\partial^2}{\partial F^2} G(0,0) = 0$$

i.e.,

$$\sigma_1^2 = \delta \frac{(G''_{21} + G''_{12})}{G'_2} - \frac{G''_{11}}{G'_2} - \delta^2 \frac{G''_{22}}{G'_2} \equiv Q(\delta, 1)$$

Note that this quadratic form $Q(\delta, 1)$ is the variance σ^2 associated with the operator $\tilde{T}(\beta) \equiv T(\beta, \delta\beta)$, to which we can apply the arguments of Section 7 (because it depends only on one variable). Let us turn first to $G(\beta_1, \beta_2)$ (slightly modified by subtraction of averages, which we can always suppose to be zero) and let us consider its nth iterate applied to the function $l(x_0, u)$:

$$G^n(\beta_1, \beta_2)\, l(x_0, u) = E_{x_0}(e^{[\beta_1 \log|M(x_n)\dots M(x_0)u|]/|u|} e^{-\beta_1(g(x_n) + \dots + g(x_0))}$$
$$\times e^{-n\beta_1(\gamma - E\log 2)} e^{\beta_2(l(x_n) + \dots + l(x_0))} e^{-n\beta_2 E\log\gamma})$$

We set $\beta_2 = \delta\beta_1$, where δ is a real, arbitrary parameter. We have

$$G^n(\beta_1, \delta\beta_1)\, l(x_0, u) = E_{x_0}(e^{[\beta_1 \log|M(x_n)\dots M(x_0)u|]/|u|} e^{-\beta_1(g(x_n) + \dots + g(x_0))}$$
$$\times e^{-n\beta_1(\gamma - E\log 2)} e^{\delta\beta_1(l(x_n) + \dots + l(x_0))} e^{-n\delta\beta_1 E\log\gamma})$$

and, in the abridged notations introduced above in this section,

$$\frac{d}{d\beta_1} G^n(\beta_1, \delta\beta_1)\, l(x_0, u)$$
$$= E_{x_0}((\log S_n - g_n - n(\gamma - E\log 2) + \delta(l_n - nE\log\gamma))$$
$$\times e^{\beta_1 \log S_n} e^{-\beta_1 g_n} e^{-n\beta_1(\gamma - E\log 2)} e^{\delta\beta_1 l_n} e^{-n\delta\beta_1 E\log\gamma})$$

and finally

$$\frac{d^2}{d\beta_1^2} G^n(0,0)\, l(x_0,u) = E_{x_0}(\log S_n - g_n - n(\gamma - E\log 2) + \delta(l_n - nE\log\gamma))^2$$

{Note that in Section 7 we had

$$\frac{d^2}{d\beta_1^2} T^n(0)\, l(x_0,u) = E_{x_0}(\log S_n - n(\gamma - E\log 2))^2\}$$

The argument of Section 7 implies that $\exists\phi \equiv \phi_\delta \in L_{\eta\phi}$ such that (cf. Proposition 7.2)

$$\sigma_\delta^2 = \sum_{x_0, v} \pi(x_0)\, v_{x_0}(v) \sum_{x_1} \pi(x_1 \mid x_0) \Bigg[\log \frac{|M(x_0)u|}{2^{(x_0)}} - \gamma + E\log 2 + \delta(l(x_0) - E\log\gamma) + \phi(x_1, M(x_0)u) - \phi(x_0, u)\Bigg]^2$$

where ϕ solves

$$\phi(x_0,u) - \sum_{x_1} \pi(x_1 \mid x_0)\, \phi(x_1, M(x_0)u) = \log \frac{|M(x_0 u)|}{2^{(x_0)}} - \gamma + E\log 2 + \delta l(x_0) - \delta E\log\gamma$$

Similarly, we have that $\sigma_\delta = 0$ if and only if $\pi \times v$-almost all (x_0, u) and $\forall n$

$$\log \frac{|M(x_n)\dots M(x_0)u|}{2^{(x_0+\dots+x_n)}} - (\gamma + E\log 2)(n+1) + \delta((l(x_0) + \dots + l(x_n)) - (n+1)\, E\log\gamma) = \phi(x_{n+1}, M(x_n)\dots M(x_0)u) - \phi(x_0, u) \qquad P^n_{x_0}\text{-a.s.}$$

On the same set of positive measure as in Section 7, we have

$$\left|\log\left(\frac{(n+3)\, e^{-n\gamma}}{(2^{2n}\varepsilon^{-n\log 2})}\right) + \delta \log(\gamma^{2n} e^{-nE\log\gamma})\right| = 2\left|\phi\left(0, \binom{2}{n+1}\right)\right|$$

Now, the l.h.s. equals $\log(n+3) + nc_1 + \delta nc_2$, whereas the r.h.s. can be bounded by $2^{5\gamma}$. This implies that (because of the logarithms) it is impossible to solve this equation for all n.

This shows that σ_δ^2 cannot be zero and consequently $(\partial^2/\partial\beta^2)F$ cannot be zero.

Definition 9.8. Let

$$f(\alpha+\delta) = \sup_{|\beta| < \hat{\beta}} \{(\alpha+\delta)\beta - F(\beta)\}$$

Corollary 9.9. The function $\alpha \to f(\alpha+\delta)$ is strictly convex in a neighborhood of zero, $f(\delta) = 0$, and $f(\alpha+\delta) < 0$ if $\alpha \neq 0$.

10. LARGE-DEVIATION PROPERTIES

We state a large-deviation theorem which we shall apply to the multifractal spectrum in Part II. Let

$$\begin{aligned} &G(\beta_1, \beta_2) \\ &\quad = \lim_{n\to\infty} \frac{1}{n} \log E_{x_0}(e^{[\beta_1 \log |M(x_n)\dots M(x_0)u|]/|u|} \\ &\qquad \times e^{-\beta_1[g(x_n)+\dots+g(x_0)-n(\gamma - E\log 2)]} e^{\beta_2[l(x_n)+\dots+l(x_0)-nE\log\beta]} \end{aligned}$$

Let $F(\beta)$ be the unique solution of $G(\beta, F(\beta)) = 0$, let $f(\alpha+\delta) = \inf_{|\beta|<\hat{\beta}} \{(\alpha+\delta)\beta - F(\beta)\}$ [i.e., $f(\alpha+\delta) = \beta(\alpha+\delta)(\alpha+\delta) - F(\beta(\alpha+\delta))$, $\beta(\alpha)$ the unique solution of $\alpha = F'(\beta(\alpha))$]. The large-deviation theorem states that

$$\frac{1}{n} \log P_{x_0}\left(\frac{1}{n} \log \frac{|M(x_n)\dots M(x_0)u|}{2^{(x_0+\dots+x_n)}} > \alpha_1, \frac{1}{n}(l(x_0)+\dots+l(x_n)) < \alpha_2\right) \xrightarrow[n\to\infty]{} \sigma(\alpha_1, \alpha_2)$$

if $\alpha_1 < \alpha_1(\max)$, $\alpha_2 > \alpha_2(\max)$, with $(\alpha_1(\max), \alpha_2(\max))$ being the point of the maximum of the convex function

$$\sigma(\alpha_1, \alpha_2) = \sup_{\beta_1, \beta_2} [\alpha_1\beta_1 + \alpha_2\beta_2 - G(\beta_1, \beta_2)]$$

We introduce the constraint $\alpha_1 = (\alpha+\delta)\,\alpha_2$ so that

$$\sigma((\alpha+\delta)\,\alpha_2, \alpha_2) = \sup_{\beta_1, \beta_2} [(\alpha+\delta)\,\alpha_2\beta_1 + \alpha_2\beta_2 - G(\beta_1, \beta_2)]$$

and, if the supremum is attained at the point

$$(\beta_1((\alpha+\delta)\,\alpha_2, \alpha_2), \beta_2((\alpha+\delta)\,\alpha_2, \alpha_2)) \equiv (\beta_1^*, \beta_2^*)$$

we have

$$\sigma((\alpha+\delta)\,\alpha_2, \alpha_2) = (\alpha+\delta)\,\alpha_2\beta_1^* + \alpha_2\beta_2^* - G(\beta_1^*, \beta_2^*)$$

so that

$$\frac{d\sigma}{d\alpha_2} = (\alpha+\delta)\,\beta_1^* + (\alpha+\delta)\,\alpha_2[(\alpha+\delta)\,\partial_1\beta_1^* + \partial_2\beta_1^*]$$

$$+\beta_2^* + \alpha_2[(\alpha+\delta)\,\partial_1\beta_2^* + \partial_2\beta_2^*]$$

$$-\frac{\partial G}{\partial\beta_1}[(\alpha+\delta)\,\partial_1\beta_1^* + \partial_2\beta_2^*] - \frac{\partial G}{\partial\beta_2}[(\alpha+\delta)\,\partial_1\beta_2^* + \partial_2\beta_2^*]$$

Combining this with the equations for the stationary point (β_1^*, β_2^*):

$$(\alpha+\delta)\,\alpha_2 = \frac{\partial G}{\partial\beta_1}(\beta_1^*, \beta_2^*), \qquad \alpha_2 = \frac{\partial G}{\partial\beta_2}(\beta_1^*, \beta_2^*)$$

we obtain

$$\sigma((\alpha+\delta)\,\alpha_2, \alpha_2) = \alpha_2[(\alpha+\delta)\,\beta_1^* + \beta_2^*] - G(\beta_1^*, \beta_2^*)$$

If we look for the point $\alpha_2(\text{critical}) = \alpha_2(c)$ such that $G(\beta_1^*, \beta_2^*) = 0$ (we shall see the meaning of this point in Part II), we obtain

$$\frac{d\sigma}{d\alpha_2}((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)) = (\alpha+\delta)\,\beta_1^* + \beta_2^* = \phi(\alpha+\delta)$$

for some function ϕ and

$$\sigma((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)) = \alpha_2(c)[(\alpha+\delta)\,\beta_1^* + \beta_2^*] = \phi(\alpha+\delta)\,\alpha_2(c)$$

Now, $G(\beta_1^*, \beta_2^*) = 0$, $\partial G/\partial\beta_2 \neq 0$ allows the inversion $\beta_2^* = \beta_2^*(\beta_1^*)$, so that

$$\phi(\alpha+\delta) = (\alpha+\delta)\,\beta_1^* + \beta_2^* = (\alpha+\delta)\,\beta_1^* + \beta_2^*(\beta_1^*)$$

i.e.,

$$\phi(\alpha+\delta) = (\alpha+\delta)\beta_1^*((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)) + \beta_2^*(\beta_1^*((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)))$$

which is precisely $(\alpha+\delta)\,\beta(\alpha+\delta) + F(\beta(\alpha+\delta))$ or, in other words, $\phi(\alpha+\delta)$ is the Legendre transform of the unique F such that $G(\beta, F) = 0$, i.e., $\phi(\alpha+\delta)$ is $f(\alpha+\delta)$, and also $\sigma((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)) = \alpha_2(c)\,f(\alpha+\delta)$, where

$$\alpha_2(c) = \frac{\partial G}{\partial F}[\beta_1^*((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)), F(\beta_1^*((\alpha+\delta)\,\alpha_2(c), \alpha_2(c)))]$$

ACKNOWLEDGMENTS

It is our pleasure to thank Pierre Collet and Philippe Bougerol. A. P. wishes to thank Thierry Huillet and Andrzej Klopotowski for fruitful discussions.

REFERENCES

1. J. Alexander and J. Yorke, The fat baker's transformation, *Ergod. Theory Dynam. Syst.* **4**:1–23 (1984).
2. J. Alexander and D. Zagier, The entropy of a certain infinitely convolved Bernoulli measure, University of Maryland preprint (1991).
3. Ph. Bougerol, Théorèmes limites pour les systèmes linéaires à coefficients markoviens, *Prob. Theory Related Fields* **78**:193–221 (1988).
4. Ph. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators* (Birkhäuser, Basel, 1985).
5. A. Bovier, *Bernoulli Convolutions as an Invariant Measure Problem* (Bonn, 1991). preprint
6. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, New York, 1975).
7. E. Cohen, H. Kesten, and C. Newman, *Random Matrices and Their Applications* (American Mathematical Society, Providence, Rhode Island, 1984).
8. P. Collet, J. Lebowitz, and A. Porzio, The dimension spectrum of some dynamical systems, *J. Stat. Phys.* **47**(5/6):609–644 (1987).
9. J. Doob, *Stochastic Processes* (Wiley, New York, 1953).
10. P. Erdös, On the smoothness properties of a family of Bernoulli convolutions, *Am. J. Math.* **62**:180–186 (1940).
11. P. Erdös, On a family of symmetric Bernoulli convolutions, *Am. J. Math.* **61**:974–976 (1939).
12. H. Furstenberg, Non-commuting random products, *Trans. Am. Math. Soc.* **108**:377–428 (1963).
13. A. Garsia, Arithmetic properties of Bernoulli convolutions, *Trans. Am. Math. Soc.* **162**:409–432 (1962).
14. A. Garsia, Entropy and singularity of infinite convolutions, *Pacific J. Math.* **13**:1159–1169 (1963).
15. Y. Guivarc'h, Quelques propriétés asymptotiques des produits de matrices aléatoires, in *Ecole d'été St Flour 1978* (Springer-Verlag, Berlin, 1980).
16. Y. Guivarc'h, Exposants caractéristiques des produits de matrices aléatoires en dépendence markovienne, in *Probability Measures on Groups* (Springer-Verlag, Berlin, 1984).
17. T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, and B. Shraiman, Fractal measures and their singularities: The characterisation of strange sets, *Phys. Rev. A* **33**:1141–1151 (1986).
18. T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1976).
19. S. Lalley, Random series in inverse Pisot powers, preprint (1993).
20. O. Lanford, Entropy and equilibrium states in classical statistical mechanics, in *Lecture Notes in Physics*, No. 20 (Springer, Berlin, 1971).
21. F. Ledrappier, Quelques propriétés des exposants caractéristiques, in *Ecole d'été St Flour 1982* (Springer-Verlag, Berlin, 1984).
22. F. Ledrappier and A. Porzio, A dimension formula for Bernoulli convolutions, *J. Stat. Phys.* **76**(5/6):1307 (1994).

23. F. Ledrappier, On the dimension of some graphs, *Contemp. Math.* **135**:285–293 (1992).
24. F. Ledrappier and L. S. Young, The metric entropy of diffeomorphisms, *Ann. Math.* **122**:540–514 (1985).
25. W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hung.* **11**:401–416 (1960).
26. E. Le Page, Théorèmes limites pour les produits de matrices aléatoires, in *Lecture Notes in Mathematics*, No. 928 (Springer, Berlin, 1984).
27. D. Placky and J. Steinbach, *Period. Math.* **6**:338–340 (1974).
28. A. Porzio, The dimension spectrum of Axiom A attractors, *J. Stat. Phys.* **58**:923 (1990).
29. D. Ruelle, Thermodynamic formalism, in *Encyclopedia of Mathematics* (Addison-Wesley, Reading, Massachusetts, 1978).
30. D. Simpelaere, The dimension spectrum of the axiom a diffeomorphysms, *J. Stat. Phys.* **76**(5/6):1359–1375 (1994).
31. Ya. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surv.* pp. 21–69 (1975).
32. B. Solomiak, On the random series $\sum \pm \lambda^n$ (an Erdös problem). preprint 1994
33. L. S. Young, Dimension, Entropy, and Lyapunov exponents, *Ergodic Theory Dynam. Syst.* **2**:109–124 (1982).